NORMAL OPERATORS AND MULTIPLIERS ON COMPLEX BANACH SPACES AND A SYMMETRY PROPERTY OF L^1-PREDUAL SPACES

BY
EHRHARD BEHRENDS

ABSTRACT
An operator T on a complex Banach space X is called *normal* if there exists an operator S such that $(1/2)(T+S)$ and $(1/2i)(T-S)$ are hermitian and $TS = ST$. We show that $T: X \to X$ is normal iff $T': X' \to X'$ is normal. Using a generalization of the principle of local reflexivity this result enables us to prove that multipliers on complex L^1-predual spaces are always normal.

Let X be a nonzero complex Banach space and T, S operators on X. We will say that S is an *adjoint* of T if both $(1/2)(T+S)$ and $(1/2i)(T-S)$ are hermitian (an operator H is called hermitian if $\|\exp(itH)\| \leqq 1$ for every $t \in \mathbf{R}$).

Using the well-known properties of hermitian elements in Banach algebras (cf. for example chapter 2 in [5]) it is easy to see that T has at most one adjoint so that we are justified in denoting such an adjoint by T^* if it exists. If T^* exists and commutes with T we say that T is *normal*.

Adjoints and normal operators have implicitly been treated in [5] in the slightly more general setting of Banach algebras.

1. THEOREM. *T is normal iff T' is normal. In this case we have* $(T^*)' = (T')^*$.

PROOF. Obviously T' is normal if T is normal. Conversely, suppose that T' is normal. We have only to show that $(T')^*$ is weak*-continuous since S is an adjoint of T which commutes with T if $S' = (T')^*$.

For the proof of this fact we combine the following two assertions:

(A) a general Fuglede–Putnam theorem is valid, i.e., if N and M are normal operators, then $NS = SM$ implies $N^*S = SM^*$ for every operator S;

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(B) an operator $S: X' \to X'$ is weak*-continuous iff S'' commutes with $P := i_{X'} \circ (i_X)' : X''' \to X'''$ (where $i_X : X \to X''$ denotes the canonical injection).

(The proof of (B) is an exercise in elementary functional analysis and for the proof of (A) one only has to note that the proof of the Fuglede–Putnam theorem in [6] for the case of operators on a Hilbert space is valid in our more general situation; it is only of importance that the norms of $\exp i(\bar{\lambda} M + \lambda M^*)$, $\exp i(\bar{\lambda} N + \lambda N^*)$ are uniformly bounded for $\lambda \in \mathbf{C}$, but this is satisfied since $\bar{\lambda} M + \lambda M^*$ and $\bar{\lambda} N + \lambda N^*$ are hermitian.)

Since T' is weak*-continuous, $T'''P = PT'''$. But T''' is normal by the first part of the proof so that $((T')^*)''P = (T''')^* P = P(T''')^* = P((T')^*)''$. Therefore $(T')^*$ is weak*-continuous.

NOTES. (1) The idea of applying (B) to the proof of the weak*-continuity of $(T')^*$ is due to G. Wodinski.

(2) We do not know whether adjoints of weak*-continuous operators on dual spaces are also weak*-continuous, i.e. whether "T admits an adjoint iff T' admits an adjoint" is valid.

In some cases the spaces X' or X'' have nicer properties than X so that it might be easier to discuss T' or T'' rather than T itself. Of course this is reasonable only if one has sufficient information about T' or T''. The following generalization of the principle of local reflexivity implies that T'' inherits much information from T.

2. THEOREM [4]. *Given a finite-dimensional space H of operators on X and finite-dimensional subspaces $E \subset X''$ and $G \subset X'$ there is, for every $\varepsilon > 0$, an isomorphism I from $F := E + \operatorname{lin} \{R''E \mid R \in H\}$ into X such that*

(i) $\| I \| \| I^{-1} \| \leqq 1 + \varepsilon$,

(ii) $I|_{E \cap X} = \mathrm{Id}|_{E \cap X}$,

(iii) $x'(Ix'') = x''(x')$ *for* $x'' \in F$, $x' \in G$,

(iv) $\| (IR'' - RI)|_E \| \leqq \varepsilon \| R \|$ *for* $R \in H$.

Thus *a strategy to prove that certain operators are normal* could be the following combination of Theorem 1 and Theorem 2:

Consider classes of operators Op which have the property that T'' is in Op whenever T is in Op (here Theorem 2 will come into play). Further, consider Banach spaces X such that every operator on X'' which is an element of Op is normal. Then every operator on X which belongs to Op is normal.

We are now going to *illustrate this strategy by an example.*

3. Definition ([1], [2]). Let $T : X \to X$ be an operator. T is called a *multiplier* if for every extreme functional p on X there is an $a_T(p) \in \mathbb{C}$ such that $p \circ T = a_T(p)p$ (i.e. if every extreme functional is an eigenvector of T').

Examples. (1) Let $X = CK$ be the space of continuous functions on a compact Hausdorff space K. Then the multipliers on X are precisely the operators $M_h : f \mapsto hf$ $(h \in X)$. This follows at once from the fact that the extreme functionals are just the mappings $f \mapsto \lambda f(k)$, where $\lambda \in \mathbb{C}$, $|\lambda| = 1$, $k \in K$.

(2) More generally, if A is any function algebra, then the multipliers on A are just the operators $M_h : f \mapsto hf$ $(h \in A)$; cf. [3].

4. Lemma. *Let T be a normal multiplier. Then T^* is also a multiplier, and $a_{T^*}(p) = \overline{a_T(p)}$ for every extreme functional p.*

Proof. We have to apply the following assertion with $S := T' : X' \to X'$:

$$(*)\qquad \begin{array}{l}\text{If } S : Y \to Y \text{ is normal and } Sy_0 = \lambda y_0 \\ \text{for some } \lambda \in \mathbb{C},\ y_0 \in Y, \text{ then } S^* y_0 = \bar{\lambda} y_0.\end{array}$$

Consider $N_\lambda := \{y \mid Sy = \lambda y\}$. Since S is normal, N_λ is invariant with respect to S^*. Thus $S^*|_{N_\lambda} = (S|_{N_\lambda})^*$. It follows that $S^*|_{N_\lambda} = (\lambda \,\mathrm{Id}_{N_\lambda})^* = \bar{\lambda}\,\mathrm{Id}_{N_\lambda}$, and this proves $(*)$.

In M-structure theory a multiplier S is called an adjoint of a multiplier T if $a_S(p) = \overline{a_T(p)}$ for every extreme p. To avoid confusion with the terminology of this paper we will call such an S a *multiplier-adjoint* of T. If a multiplier T admits a multiplier-adjoint S then T is obviously normal and $S = T^*$. The preceding lemma just asserts that the adjoint of a normal multiplier is the multiplier-adjoint so that both definitions of adjoints are consistent. We do not know, however, whether normality is essential, i.e. whether adjoints of multipliers are also multipliers.

It depends on the geometry of X whether every multiplier is normal. For example, if $X = CK$, we have $(M_h^*) = M_{\bar{h}}$ for every multiplier M_h, and $M_{\bar{h}}$ is a multiplier-adjoint of M_h.

(More generally, if T is a multiplier on a space X which can isometrically be embedded as a self-adjoint subspace of a CK-space then a multiplier-adjoint of T can be defined by $T^*x := \overline{(T\bar{x})}$.)

On the other hand, if X is the disk algebra, then the only normal multipliers are the constant multiples of the identity operator.

The following corollary solves a problem from M-structure theory of complex Banach spaces (cf. p. 71 in [2]).

5. Corollary. *Let T be a multiplier on a complex Banach space X. If T' can be arbitrarily well approximated by linear combinations of L-projections then T is normal.*

(An L-projection on X' is a linear projection $E : X' \to X'$ such that $\| p \| = \| Ep \| + \| p - Ep \|$ for every $p \in X'$.)

Proof. Let $C(X')$ be the closed linear span of all L-projections on X'. It is well-known that $C(X')$ is isometrically and algebra isomorphic with the space of continuous functions on a compact Hausdorff space (see e.g. prop. 1.16 in [2]) so that every element of $C(X')$ is normal. Thus our assertion follows from Theorem 1.

We now are going to show that multipliers on L^1-predual spaces are normal (a problem which has been the starting point of the present investigations). As a preparation we need

6. Lemma. *Let $T : X \to X$ be an operator. Then T is a multiplier iff T'' is a multiplier.*

Proof. Every extreme functional on X can be extended as an extreme functional to X''. This implies that T is a multiplier if T'' is a multiplier. Now suppose that T is a multiplier. By [2], theorem 3.3 there is a $\lambda > 0$ such that

$$(*)\qquad \begin{array}{l} \text{if } B = B(x_0, r) \text{ is an open ball and } x \in X \text{ is an element} \\ \text{such that } \{\mu x \mid |\mu| \leqq \lambda\} \subset B, \text{ then } \| x_0 - Tx \| < r, \end{array}$$

and by the same theorem we have to show that

$$(**)\qquad \begin{array}{l} \text{if } B = B(x''_0, r) \text{ is an open ball in } X'', \text{ then} \\ \{\mu x'' \mid |\mu| \leqq \lambda\} \subset B \text{ implies that } \| x''_0 - T''x'' \| < r. \end{array}$$

Let such a ball and x'' be given. We apply Theorem 2 with $H := \operatorname{lin}\{T, \mathrm{Id}\}$, $E := \operatorname{lin}\{x''_0, x''\}$, $F := \{0\}$ (ε will be specified later). With $x_0 := Ix''_0$, $x := Ix''$ we get

$$\begin{aligned} \| x_0 - \mu x \| &\leqq (1 + \varepsilon) \| x''_0 - \mu x'' \| \\ &\leqq (1 + \varepsilon) r' \qquad \text{for every } \mu \text{ with } |\mu| \leqq \lambda \end{aligned}$$

so that $\| x_0 - Tx \| \leqq (1+\varepsilon)r'$ by (*). (Here r' is any number such that $0 < r' < r$ and $\max_{|\mu| \leqq \lambda} \| x_0'' - \mu x'' \| \leqq r'$.)

We have used the fact that I can be chosen with $\| I \| \leqq 1+\varepsilon$ (and $\| I^{-1} \| \leqq 1+\varepsilon$) and not only $\| I \| \, \| I^{-1} \| \leqq 1+\varepsilon$; this is a consequence of $I|_{E\cap X} = \mathrm{Id}|_{E\cap X}$ since E can be enlarged if necessary such that $E \cap X \neq \{0\}$.

It follows that

$$\begin{aligned} \| x_0'' - T''x'' \| &\leqq (1+\varepsilon) \| x_0 - IT''x'' \| \\ &= (1+\varepsilon) \| x_0 - IT''x'' + TIx'' - TIx'' \| \\ &\leqq (1+\varepsilon)(\| x_0 - Tx \| + \| (TI - IT'')x'' \|) \\ &\leqq (1+\varepsilon)((1+\varepsilon)r' + \varepsilon \, \| T \| \, \| x'' \|), \end{aligned}$$

and this expression is less than r if ε is sufficiently small.

7. THEOREM. *Let X be an L^1-predual space. Then every multiplier on X is normal.*

PROOF. X'' is a CK-space, and we have already noted that multipliers on CK-spaces are normal. Therefore the theorem is a consequence of Theorem 1 and Lemma 6.

NOTE. Roughly speaking the theorem expresses the following *symmetry property* of complex L^1-predual spaces X: If a multiplier can be defined on X then in its definition only such expressions occur which are invariant with respect to complex conjugation. A simple example should illustrate this:

If X is the space $\{(x_n) \mid (x_n) \in c, x_n \to \mu x_1\}$ (where $\mu \in \mathbf{C}, 0 < |\mu| \leqq 1$) then the multipliers on this L^1-predual space are precisely the operators which have the form $(x_n) \mapsto (\lambda_n x_n)$, where

$$(*) \qquad (\lambda_n) \in c, \qquad \lambda_n \to \lambda_1.$$

If T is defined by $(x_n) \mapsto (\lambda_n x_n)$, then T^* is just the operator $(x_n) \mapsto (\bar{\lambda}_n x_n)$, and this definition makes sense since both conditions in (*) are invariant under complex conjugation.

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I. Mathematisches Institut der Freien Universität Berlin
Arnimallee 2–6 ·
D 1000 Berlin 33, FRG